

Straightening out planar poly-line drawings

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Abstract. This paper addresses the following question: Given a planar poly-line drawing of a graph, can we “straighten it out”, i.e., convert it to a planar straight-line drawing, while keeping some features unchanged? We show that any y -monotone poly-line drawing can be straightened out while maintaining y -coordinates and height. The width may increase much, but we also show that on some graphs exponential width is required if we do not want to increase the height. Likewise y -monotonicity is required: there are poly-line drawings (not y -monotone) that cannot be straightened out while maintaining the height. We give some applications of our result.

Keywords: Graph drawing; poly-line drawing; straight-line drawing

1 Introduction

Let $G = (V, E)$ be a simple graph with $n = |V|$ vertices and $m = |E|$ edges. We assume that G is *planar*, i.e., it can be drawn without crossings. Almost by definition G has a planar *poly-line drawing*, i.e., a drawing where vertices are points, edges are drawn as sequences of contiguous line segments and there are no crossings. For if any part of a planar drawing of G uses curved lines, then these can be approximated by polygonal curves.

However, for ease of readability and storing the drawing, one prefers a *straight-line drawing*, i.e., a drawing where vertices are points and edges are straight-line segments between their endpoints. It was known for a long time that any planar graph has a straight-line drawing [7,12,13], even in an $O(n) \times O(n)$ -grid [9,11]. Many improvements have been developed since, see for example [5,10].

This paper addresses the following question: Given a planar poly-line drawing Γ of a graph G , can we “straighten it out”? By the above, there exists a straight-line drawing of G , but can we create a straight-line drawing Γ' of G that is similar to Γ in some sense? For example, for some applications (we list a few in Section 6) it is much easier to create a poly-line drawing; can it be converted to a straight-line drawing without losing some key features?

The answer to our main question obviously depends on what is meant by “similar”. One possible way to define this would be to request that the relative order of vertices stays the same, i.e., v is to the left / below w in Γ if and only

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if it is in Γ' . However, one quickly sees that then not all poly-line drawings can be straightened out, see the drawing of K_4 in Figure 1(left).

The main result (Theorem 1) of this paper is that if a poly-line drawing is y -monotone, then it can be converted into a straight-line drawing such that y -coordinates of vertices are unchanged, and in particular the height of the drawing is unchanged. This result has some applications that we list. Most importantly, it allows to derive some height-bounds for drawing styles for which we are not aware of any direct proof, and it allows to formulate some NP-hard graph drawing problems as integer programs.

We also give graphs where two limitations of Theorem 1 cannot be overcome: It is not possible to remove the “ y -monotone” condition without increasing the height, and it is not possible to keep the width to less than exponential without increasing the height.

Historical note: At WAOA’12 [2], we claimed a result (about straightening out so-called flat visibility representations) that would easily imply Theorem 1. Unfortunately the algorithm in [2] is incorrect since the resulting drawing may not be planar; we review the algorithm and show the error in the appendix.

2 Preliminaries

Throughout this paper G denotes a planar graph, Γ denotes a planar poly-line drawing of G , and Γ' denotes the straight-line drawing of G that we try to construct. Recall that in Γ edges are drawn as sequences of contiguous line segments; a point of transition between the line segments is called a *bend*. Γ is called *y -monotone* if for any edge, the y -coordinates monotonically increases as we walk from one end to the other. Horizontal segments are allowed.

We call a drawing a *grid-drawing* if vertices and bends have integer coordinates. A grid drawing is said to have *width* w and *height* h if (possibly after translation) vertices and bends are placed on the $[1, w] \times [1, h]$ -grid. The height is thus measured by the number of *rows*, i.e., horizontal lines with integer y -coordinates that are occupied by the drawing.

Our transformation will be such that Γ and Γ' have the same y -coordinates, i.e., any vertex has the same y -coordinate in both drawings. Since we give transformations only for y -monotone drawings, this implies that an edge e intersects row r in Γ if and only if it intersects row r in Γ' .

Our transformation actually achieves a stronger property: Γ and Γ' have the same *left-to-right order of edges and vertices in each row*. This means that if we enumerate in row r from left to right the segments s_1, s_2, s_3, \dots that are occupied by Γ (i.e., each s_i is either a point containing a vertex, or a bend of an edge, or a point where an edge segment crosses row r , or a line segment where an edge is routed along r), and if we similarly enumerate the segments s'_1, s'_2, s'_3, \dots occupied by Γ' in row r , then for all i segments s_i and s'_i belong to the same vertex or the same edge.

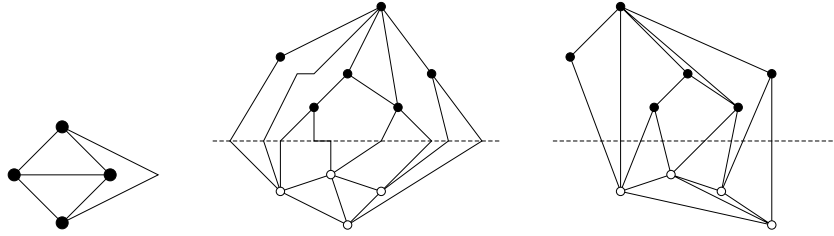


Fig. 1. y -monotone poly-line drawings. (Left) K_4 cannot be straightened out while maintaining relative x - and y -coordinates. (Middle) An HH-drawing (see Section 6). (Right) The HH-drawing straightened out while maintaining y -coordinates and left-to-right-orders in each row.

Lemma 1. *Let Γ, Γ' be y -monotone poly-line drawings of the same graph G . If Γ is planar, and Γ' has the same y -coordinates and the same left-to-right-orders in each row as Γ , then Γ' is also planar, with the same planar embedding and outer-face.*

Proof. (Sketch) If edges e and e' crossed in Γ' , but not in Γ , then in the row above or below the point of the crossing the left-to-right orders are not the same in Γ and Γ' . \square

3 Creating straight-line drawings

In this section, we show how to convert any planar y -monotone poly-line drawing into a planar straight-line drawing.

3.1 Straightening out triangulated drawings

We first show such a transformation for a *triangulated graph*, i.e., for a graph where all faces including the outer-face are triangles. It will be helpful to direct an edge (v, w) as $v \rightarrow w$ if $y(v) < y(w)$, and replace horizontal edges by two directed edges in opposite directions. Recall that an *inner vertex* is a vertex that is not on the outer-face.

Observation 1 *Let Γ be a planar y -monotone poly-line drawing of a triangulated graph G . Then $\text{indeg}(v) \geq 1$ and $\text{outdeg}(v) \geq 1$ for every inner vertex.*

Proof. (Sketch) If $\text{indeg}(v) = 0$ for an inner vertex v , then the face “below” v would have degree ≥ 4 since edges are drawn y -monotonically. \square

The conversion into a straight-line drawing will happen by repeatedly contracting one vertex that has properties that allow to re-insert it later. The existence of such a vertex is proved in the following lemma.

Lemma 2. *Let Γ be a planar y -monotone poly-line drawing of a triangulated graph G . Then there exists an inner vertex v such that*

- v is incident to a horizontal edge, or
- $\text{indeg}(v) = 1$, and $\text{indeg}(w) \geq 2$ for all vertices w with $v \rightarrow w$, or
- $\text{outdeg}(v) = 1$, and $\text{outdeg}(w) \geq 2$ for all vertices w with $w \rightarrow v$.

Proof. Since the outer-face contains three vertices, one of the bottom and the top row (say the bottom row) contains only one vertex. Let v_1 be the inner vertex that minimizes the y -coordinate among inner vertices, breaking ties arbitrarily.

We know $\text{indeg}(v_1) \geq 1$ since v_1 is not on the outer-face. At most one vertex is strictly below v_1 by choice of v_1 and since the bottom row contains only one vertex. If $\text{indeg}(v_1) \geq 2$, then v_1 hence has an incident horizontal edge and we are done. Likewise we are done if $\text{indeg}(v_1) = 1$ and $\text{indeg}(w) \geq 2$ for all $v_1 \rightarrow w$. So assume that $\text{indeg}(w) = 1$ for some $v_1 \rightarrow w$, and set $v_2 := w$. Observe that v_2 is an inner vertex, since any outer-face vertex x with $y(x) \geq y(v)$ has another incoming edge from the vertex on the bottom row. Repeat with v_2 : either it is incident to a horizontal edge, or all its successors have indegree ≥ 2 , or it has a successor with indegree 1, which we set to be v_3 . Repeat. Eventually this must find a suitable vertex, since otherwise there would be an infinite sequence v_1, v_2, v_3, \dots with $y(v_1) < y(v_2) < \dots$. \square

The proof of the following lemma gives the main part of the algorithm to straighten out y -monotone poly-line drawings:

Lemma 3. *Let Γ be a planar y -monotone poly-line drawing of a triangulated graph G . Then there exists a planar straight-line drawing Γ' of G that has the same y -coordinates and the same left-to-right orders as Γ .*

Proof. If $n = 3$, then G is a triangle and the result is easily shown. Now assume $n \geq 4$. We have three cases:

Case 1: G contains a *separating triangle*, i.e., a triangle $\{u, v, w\}$ such that there are other vertices both inside and outside $\{u, v, w\}$.

Let G_0 be the graph induced by $\{u, v, w\}$ and all vertices outside it, and let G_1 be the graph induced by $\{u, v, w\}$ and all vertices inside it. Recursively apply the lemma to these two graphs, with the drawing the one induced by Γ , to obtain the straight-line drawings Γ_0 of G_0 and Γ_1 of G_1 . The transformation preserves y -coordinates and left-to-right-orders for the two copies of $\{u, v, w\}$. After a horizontal translation and a horizontal shear, the drawing Γ_1 can hence be inserted into the face $\{u, v, w\}$ in the drawing Γ_0 to give a straight-line drawing of G . All properties are easily verified.

Case 2: Γ contains an inner vertex v with an incident horizontal edge (v, w) . Let $w, z_1, x_1, \dots, x_d, z_2$ be the neighbours of v in clockwise order; note that z_1, z_2 are also neighbours of w since the faces incident to (v, w) are triangles. If any x_i is a neighbour of w then $\{v, w, x_i\}$ is a separating triangle and we are done by Case 1. So assume that v, w only have z_1, z_2 as common neighbours. Contract v into w and delete the duplicate copies of (w, z_1) and (w, z_2) ; this then yields a

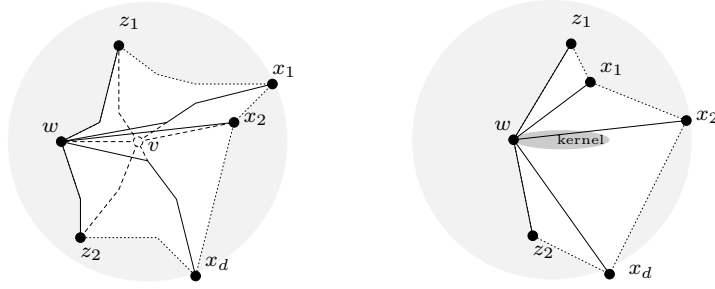


Fig. 2. Case 2: An inner vertex v has a horizontal edge (v, w) . (Left) Contracting v into w and how to re-route edges. (Right) We can insert v in the kernel.

simple graph G_0 . Create a poly-line drawing Γ_0 of G_0 by routing the edge from x_i to w along the route of (x_i, v) until the row above/below v , and then to w . See Figure 2(left).

By induction transform Γ_0 into a planar straight-line drawing Γ'_0 with the same y -coordinates and same left-to-right-orders. Deleting the edges (w, x_i) from Γ'_0 leaves a polygon P defined by $w, z_1, x_1, \dots, x_d, z_2, w$ into which we must insert v . The boundary of P consists of edges of G_0 since G_0 is triangulated. Vertex w is adjacent to all vertices of P , hence P is star-shaped with w in its kernel. Since (w, x_i) (for $i = 1, \dots, d$) was drawn in Γ'_0 without overlapping, it belongs to the interior of P . Therefore the kernel actually contains an open region around w . Inside this open region we can find a point $p(v)$, with y -coordinate $y(v)$ and to the right of w , where we place v . Since v is in the kernel of P , we can then connect all its edges without introducing crossings and obtain Γ' . By construction the y -coordinate of v is the same as in Γ , and one easily verifies that left-to-right-orders are the same in all rows the clockwise order of edges at v is correct in Γ' .

Case 3: None of the above. By Lemma 2, there exists an inner vertex v with $\text{indeg}(v) = 1$ and $\text{indeg}(w) \geq 2$ for all $v \rightarrow w$. (The case of a vertex v with $\text{outdeg}(v) = 1$ and $\text{outdeg}(w) \geq 2$ for all $w \rightarrow v$ is symmetric.)

Let u, x_1, \dots, x_d be the neighbours of v in clockwise order, where $u \rightarrow v$ is the unique incoming edge of v . Since Case 2 does not apply, v has no incident horizontal edge and so $y(u) < y(v) < \min\{y(x_1), y(x_d)\}$. Assume for contradiction that $y(x_{i-1}) > y(x_i) < y(x_{i+1})$ for some $1 < i < d$, i.e., that there exists a local minimum (with respect to y -coordinates) among the neighbours above v . Then $\text{indeg}(x_i) = 1$, since the incoming edges are consecutive in the cyclic order, but the edges before and after $v \rightarrow x_i$ at x_i are both outgoing. This contradicts the choice of v . It follows that

$$y(v) < y(x_1) \leq y(x_2) \leq \dots \leq y(x_m) \geq y(x_{m+1}) \geq \dots \geq y(x_d) > y(v)$$

for some $1 \leq m \leq d$. See Figure 3(left). Let G_0 be the graph obtained from G by contracting v into u and deleting the duplicate copies of (u, x_1) and (u, x_d) that result. This does not create multiple edges since the only common neighbours

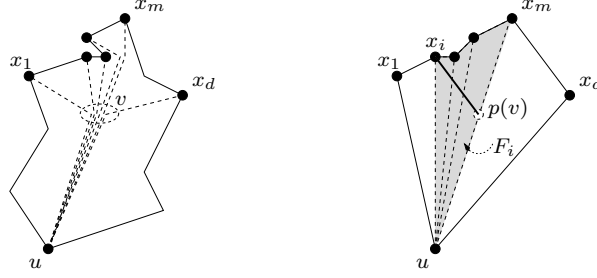


Fig. 3. Case 3: $\text{indeg}(v) = 1$ and $\text{indeg}(w) \geq 2$ for $v \rightarrow w$. (Left) Contracting v into u . (Right) Point $p(v)$ can see x_i .

of u and v are x_1 and x_d (otherwise we'd have a separating triangle and be in Case 1.) Let Γ_0 be the poly-line drawing obtained from Γ by deleting v and then inserting the above edges; this can be done by drawing (u, x_i) near the drawings of (u, v) and (v, x_i) for $i = 1, \dots, d$.

By induction we can convert Γ_0 into a planar straight-line drawing Γ'_0 with the same y -coordinates and left-to-right-orders. Delete the added edges from Γ'_0 , and let $p(v)$ be the point where the edge (u, x_m) intersected the row with y -coordinate $y(v)$. We claim that point $p(v)$ can see all of u, x_1, \dots, x_d :

- Clearly $p(v)$ can see u and x_m since it is placed on where edge (u, x_m) was drawn as a straight line.
- For any $1 \leq i < m$ consider the polygon F_i defined by the union of the faces $\{u, x_i, x_{i+1}\}, \dots, \{u, x_{m-1}, x_m\}$ in Γ'_0 . Since these were faces, and we deleted the edges $(u, x_{i+1}), \dots, (u, x_m)$, polygon F_i has an empty interior. Since $y(p(v)) = y(v) < y(x_i)$, while $y(x_i) \leq y(x_{i+1}) \leq \dots \leq y(x_m)$, the line segment from $p(v)$ to x_i is inside F_i , so $p(v)$ can see x_i .
- Similarly we prove that $p(v)$ can see any x_i with $m < i \leq d$.

Hence placing v at $p(v)$ gives a planar drawing that respects the y -coordinate and planar embedding at v . \square

One interesting consequence of this proof is that for any y -monotone planar poly-line drawing of height h , there exists a pair of vertices such that contracting them yields again a planar y -monotone poly-line drawing of height h . This contrasts with a result in [8] that having a planar drawing of height h is *not* always preserved under contracting (arbitrary) edges.

3.2 Triangulating y -monotone poly-line drawings

For space reasons we only sketch here how to triangulate y -monotone poly-line drawings; the appendix gives all details. As a first step, add one new row each at the top and the bottom. Add one new vertex in the new bottom row and two new vertices in the new top row, and connect them, using y -monotone curves, with a triangle that encloses the entire drawing.

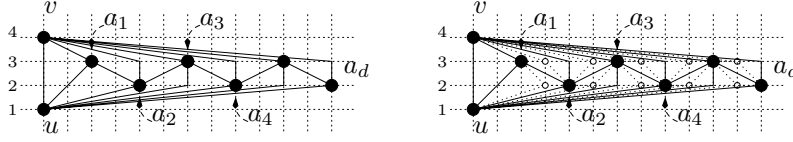


Fig. 4. (Left) A planar graph. (Right) Inserting vertices into inner faces.

As next step, ensure that every inner vertex v has a neighbour w with $y(w) > y(v)$. If v has none, then go upward from v until we hit an edge e (this must happen since v is not on the outer-face). Let w be the head of e , and note that we can route the new edge (v, w) by going upward from v and then going parallel to e . Similarly ensure that each inner vertex v has a neighbour with a smaller y -coordinate.

Now every inner face f is drawn y -monotone, with the only horizontal edges on f at its minimum or maximum y -coordinate. One can argue that under this restriction *any* two non-adjacent vertices of f can be connected with a y -monotone path inside f . Hence we can triangulate f in any way that does not insert multiple edges and eventually obtain a y -monotone poly-line drawing of a triangulated graph.

Convert the drawing into a straight-line drawing as in Lemma 3, then delete the added edges and vertices. The added rows are then empty (since y -coordinates are preserved) and so can also be deleted. So we have:

Theorem 1. *Any planar y -monotone poly-line drawing Γ can be transformed into a planar straight-line drawing Γ' with the same y -coordinates and the same left-to-right orders in each row.*

The proof of this theorem is algorithmic, and clearly leads to a quadratic-time algorithm. Reducing this run-time remains an open problem. It is not hard to build a data structure to find a vertex v as in Lemma 2 in amortized constant time, but it is not clear how we can test in less than linear time per vertex v whether v belongs to a separating triangle, because the graph changes due to the edge contractions.

4 Optimal height means exponential width

While our transformation in Theorem 1 keeps the height intact, the width can increase dramatically. Our construction does not give a grid drawing, but it is clear that with minor modifications one can achieve rational coordinates, hence integer coordinates after horizontal scaling. We do not know upper bounds on these grid coordinates, but we can argue that they are at least exponential for some graphs. To do so, we first study special drawings of one graph:

Lemma 4. *Let G be the graph shown in Figure 4(left). Then any planar straight-line drawing Γ that respects the y -coordinates and left-to-right-orders of Figure 4 has width at least $\frac{1}{3}2^{n-1}$.*

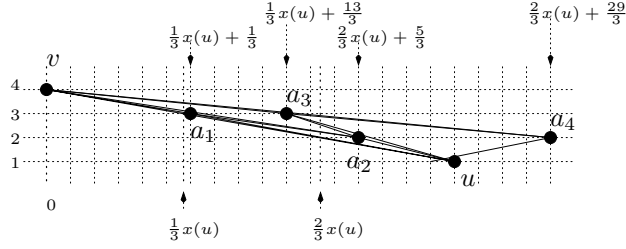


Fig. 5. Computing the required width in a straight-line drawing.

Proof. Denote by $x(w)$ the x -coordinate of vertex w in drawing Γ . We will assume that $x(v) \leq x(u)$; the other case is proved similar and in fact gives an even larger width bound. After possible translation, we may also assume $x(v) = 0$. We will show by induction on i that

$$x(a_{2i-1}) \geq \frac{1}{3}(x(u) + 2^{2i}) - 1 \quad \text{and} \quad x(a_{2i}) \geq \frac{1}{3}(2x(u) + 2^{2i+1}) - 1$$

for $i \geq 1$; this implies the result since the width is then $x(a_d) - x(v) + 1 \geq \frac{1}{3}(x(u) + 2^{d+1}) \geq \frac{1}{3}2^{d+1} = \frac{1}{3}2^{n-1}$.

Consider vertex a_1 , which is placed on row 3. The straight-line segment (u, v) crosses row 3 at x -coordinate $\frac{1}{3}x(u)$, and a_1 must be to the right of that, so

$$x(a_1) \geq \lfloor \frac{1}{3}x(u) \rfloor + 1 \geq \frac{1}{3}x(u) + \frac{1}{3} = \frac{1}{3}(x(u) + 2^2) - 1$$

as desired. Now consider vertex a_{2i} for $i \geq 1$. The line segment of edge (a_{2i}, v) crosses row 3 at x -coordinate $\frac{1}{2}x(a_{2i})$. Since left-to-right-orders are preserved, this crossing must be to the right of a_{2i-1} , therefore $x(a_{2i-1}) < \frac{1}{2}x(a_{2i})$ or $2x(a_{2i-1}) < x(a_{2i})$. By integrality therefore

$$x(a_{2i}) \geq 2x(a_{2i-1}) + 1 \geq \frac{1}{3}(2x(u) + 2 \cdot 2^{2i}) - 2 + 1$$

as desired. Finally consider vertex a_{2i+1} for $i \geq 3$. The line segment of edge (a_{2i+1}, u) crosses row 2 at x -coordinate $(x(u) + x(a_{2i+1}))/2$. Since left-to-right-orders are preserved, this crossing must be to the right of a_{2i} , therefore $x(a_{2i}) < (x(u) + x(a_{2i+1}))/2$ or $2x(a_{2i}) - x(u) < x(a_{2i+1})$. By integrality therefore

$$x(a_{2i+1}) \geq 2x(a_{2i}) - x(u) + 1 \geq \frac{1}{3}(4x(u) + 2 \cdot 2^{2i+1}) - 2 - x(u) + 1 = \frac{1}{3}(x(u) + 2^{2i+2}) - 1$$

as desired. \square

Theorem 2. *There exists a graph H that has a planar straight-line drawing on four rows, but any planar straight-line drawing on four rows has width at least $\frac{1}{3}2^{n/3}$.*

Proof. The graph H is obtained by taking the graph G from Figure 4(left) with $d \geq 11$ and inserting into each inner face except $\{u, v, a_1\}$ a new vertex adjacent to the three vertices of the face. Note that H is triangulated and has $3d$ vertices. It has a y -monotone poly-line drawing on four rows (see Figure 4(right)), and hence by Theorem 1 also a straight-line drawing on 4 rows.

Let Γ_H be an arbitrary planar straight-line drawing of H that uses four rows. Let Γ_G be the induced planar straight-line drawing of G . The goal is to show that $\Gamma_G - a_d$ satisfies the conditions of Lemma 4.

Claim: *Edge (u, v) connects the top and bottom row.* For observe that triangle $\{u, v, a_{3i}\}$ separates triangle $\{a_{3i-2}, a_{3i-1}, x\}$ from triangle $\{a_{3i+1}, a_{3i+2}, x'\}$, where $i = 1, 2, 3$ and x, x' are suitable vertices from $H - G$. Regardless of the choice of outer-face hence triangle $\{u, v, a_{3i}\}$ surrounds another triangle and must contain vertices in rows 1 and 4 by y -monotonicity. If (say) row 1 contains neither u nor v , then it must hence contain a_3, a_6 and a_9 , which means that we can construct a planar drawing of $K_{3,3}$ by adding another vertex in row 0 and connecting it to a_3, a_6, a_9 . This is impossible, so one of $\{u, v\}$ is in row 1 and the other in row 4.

If there were vertices both left and right of edge (u, v) , then $\{u, v\}$ would be a cutting pair, which contradicts that H is triangulated. So there are no vertices to one side of (u, v) . After possible horizontal flip of the drawing, renaming of $\{u, v\} \rightarrow \{v, u\}$, and renaming $\{a_1, \dots, a_d\} \rightarrow \{a_d, \dots, a_1\}$, we may assume that u is on row 1, v is on row 4, the remaining vertices are to the right of edge (u, v) , and the outer-face is $\{u, v, a_d\}$. i.e., the same as in Figure 4

Claim: *For any $1 \leq i < d$, vertex a_i is not on row 1 or 4.* For if it were on row 1, then edge (u, a_i) would be horizontal. Edge (u, a_{i+1}) comes after (u, a_i) in clockwise order around u in G , but is (by the above) to the right of u , which is impossible since there is no lower row. Similarly a_i is not on row 4 due to edges (v, a_i) and (v, a_{i+1}) .

Claim: *For any $1 \leq i < d$, vertices a_i and a_{i+1} are on different rows.* For assume a_i and a_{i+1} are both on row 2 (the case of row 3 is symmetric). Then triangle $\{u, a_i, a_{i+1}\}$ is drawn on two adjacent rows and hence has no grid-point in its interior, contradicting that in H there exists a vertex inside $\{u, a_i, a_{i+1}\}$.

If necessary, flip Γ_H upside down and rename $\{u, v\} \rightarrow \{v, u\}$ so that a_1 is on row 3. Therefore a_2 is on row 2, a_3 is on row 3, and generally a_i is on row 2 for $i < d$ even and on row 3 for $i < d$ odd. Hence $\Gamma_G - a_d$ is drawn with exactly the y -coordinates as in Figure 4. It also has the same left-to-right-orders, since the planar embedding is unique. By Lemma 4, therefore $\Gamma_G - a_d$ has width at least $\frac{1}{3}2^d = \frac{1}{3}2^{n/3}$. Γ_H can have no smaller width, which proves the theorem. \square

By the first of these claims, the graph requires 4 rows in *any* straight-line drawing. As a consequence of Theorem 2, we hence have:

Corollary 1. *There exists a planar graph such that any planar straight-line drawing that has optimal height has exponential area.*

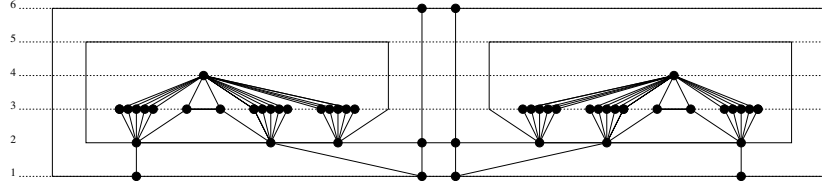


Fig. 6. A graph that can be drawn on 6 rows, but not if edges must be y -monotone.

5 Optimal height for poly-line vs. straight-line

Theorem 1 required y -monotonicity of the poly-line drawing. One can show that this condition cannot be dropped unless we allow an increase in height and a change of y -coordinates.

Theorem 3. *There exists a graph with a planar poly-line drawing on 6 rows that has no planar y -monotone poly-line drawing on 6 rows.*

The graph and its poly-line drawing on 6 rows are shown in Figure 6, and details of the proof are in the appendix. Since straight-line drawings are y -monotone poly-line drawings, we hence have that poly-line drawings are sometimes better than straight-line drawings:

Corollary 2. *There exists a graph with a planar poly-line drawing on 6 rows that has no straight-line drawing on 6 rows.*

6 Applications

We give a few applications of Theorem 1.

HH-drawings: In a previous paper [3] we studied *HH-drawings*, where we are given a planar graph G with a vertex partition $V = A \cup B$, and we would like to draw G such that all vertices in A have positive y -coordinates and all vertices in B have negative y -coordinates. See also Figure 1. We gave necessary and sufficient conditions for the existence of HH-drawings that were y -monotone poly-line drawings. We also argued that straight-line HH-drawings required exponential area for some graphs, but we were not able to actually construct straight-line HH-drawings. With Theorem 1, this missing link has now been added, because the y -monotone poly-line HH-drawing can be converted to a straight-line drawing, and it is still an HH-drawing since y -coordinates have not been changed. In particular we hence have:

Theorem 4. *Any planar bipartite graph has a planar straight-line HH-drawing.*

Drawing outer-planar graphs with small height: An *outer-planar graph* is a planar graph that can be drawn such that all vertices are on the outer-face.

A *flat visibility representation* is an assignment of disjoint horizontal segments to vertices and disjoint horizontal or vertical segments to edges so that edges touch their endpoints and do not intersect any other vertices. In [2] we showed that any 2-connected outer-planar graph G has a flat visibility representation of height at most $4pw(G)$, where the *pathwidth* $pw(G)$ is a graph parameter that is a lower bound on the height of any planar drawing [8].

By a result of Babu et al. [1], we can add edges to any outer-planar graph G to obtain a maximal outer-planar graph G' with pathwidth in $O(pw(G))$. A simple exercise shows that flat visibility representations can be converted into y -monotone poly-line drawings of the same height. Theorem 1 hence implies:

Theorem 5. *Every outer-planar graph G has a planar straight-line drawing of height $O(pw(G)) \subseteq O(\log n)$.*

Integer programming formulations: In a recent paper, we developed integer program (IP) formulations for many graph drawing problems where vertices and edges are represented by axis-aligned boxes [4]. By adding some constraints, one can force that edges degenerate to line segments and vertices to horizontal line segments. In particular, it is easy to create an IP that expresses “ G is drawn as a flat visibility representation”, using $O(n^3)$ variables and constraints.

It is quite easy to show that every flat visibility representation can be converted into a y -monotone poly-line drawing, and vice versa, without changing y -coordinates. By Theorem 1 we hence have:

Theorem 6. *A graph G has a planar straight-line drawing of height h if and only if it has a planar flat visibility representation of height h .*

It is very easy to encode the height in the IP formulations of [4]. Therefore:

Corollary 3. *There exists an integer program with $O(hn^2)$ variables and constraints to test whether a graph G has a planar straight-line drawing of height h .*

While an algorithm was already known to test whether G has a planar drawing of height at most h [6], its rather large run-time of $O(2^{32h^3} \text{poly}(n))$ means that solving the above integer program might well be faster in practice.

7 Conclusion and open problems

In this paper, we studied how to transform planar poly-line drawings into straight-line drawings. In particular we showed that y -monotone poly-line drawings are no more powerful (with respect to the height) than straight-line drawings, because any planar y -monotone poly-line drawing can be transformed into a planar straight-line drawing with the same height. If we drop “ y -monotone” then we can argue that poly-line drawings are sometimes truly better (with respect to height) than straight-line drawings.

We also demonstrated some applications of our height-preserving transformations, especially for obtaining drawings of small height.

Our main open problem concerns the width. If we want to keep the height exactly the same, then exponential width is required for the example in Figure 4. But is it possible to make the width polynomial while keeping the height asymptotically the same? Generally, what is the width of the construction in Theorem 1, and how does it depend on the height?

Also, the graph in Figure 10 can easily be drawn with y -monotone curves if we use 7 rows. Can every poly-line drawing (not necessarily y -monotone) be converted into a straight-line drawing with asymptotically the same height?

Finally, are there other invariants that one can maintain while “straightening out” poly-line drawings, at least under some restrictions such as y -monotonicity?

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A The algorithm in [2]

Recall that a *flat visibility representation* is an assignment of disjoint horizontal segments to vertices and disjoint horizontal or vertical segments to edges so that edges touch their endpoints and do not intersect any other vertices. (In the figures below, we show vertices thickened into boxes for ease of readability.) In [2], we claimed the following result.

Theorem 7. *Any planar flat visibility representation Γ can be transformed into a planar straight-line drawing Γ' with the same y -coordinates.*

While the result is correct (one can easily transform flat visibility representation into a y -monotone poly-line drawing and then apply Theorem 1), the simple algorithm that we gave for it in [2] unfortunately was incorrect. This section reviews the algorithm and gives the counter-example.

A.1 The algorithm

Assume that Γ is a planar flat visibility representation. For any vertex v , use $x_l(v)$, $x_r(v)$ and $y(v)$ to denote leftmost and rightmost x -coordinate and (unique) y -coordinate of the line segment that represents v in Γ . Use $X(v)$ and $Y(v)$ to denote the (to-be-determined) coordinates of v in the straight-line drawing Γ' that we construct. For any vertex set $Y(v) = y(v)$, hence y -coordinates are the same.

Let v_1, \dots, v_n be the vertices sorted by $x_l(\cdot)$, breaking ties arbitrarily. For each vertex v_i , let the *predecessors* of v_i be the neighbours of v_i that come earlier in the order v_1, \dots, v_n . The algorithm determines $X(\cdot)$ for each vertex by processing vertices in order v_1, \dots, v_n and expanding the drawing Γ'_{i-1} created for v_1, \dots, v_{i-1} into a drawing Γ'_i of v_1, \dots, v_i .

Suppose $X(v_g)$ has been computed for all $g < i$ already. To find $X(v_i)$, determine lower bounds for it by considering all predecessors of v_i and taking the maximum over all of them. One lower bound for $X(v_i)$ is that it needs to be to the right of anything in row $y(v_i)$. Thus, if Γ'_{i-1} contains a vertex or part of an edge at point $(X, y(v_i))$, then $X(v_i) \geq \lfloor X \rfloor + 1$ is required.

Next consider any predecessor v_g of v_i with $y(v_g) \neq y(v_i)$. Since v_g and v_i are not in the same row, they must see each other vertically in Γ , which means that $x_r(v_g) \geq x_l(v_i)$. See also Figure 7. So if in Γ any vertex v_k exists to the right of v_g in row $y(v_g)$, then $x_\ell(v_k) > x_r(v_g) \geq x_\ell(v_i)$, which implies that v_k has not yet been added to Γ'_{i-1} . So vertex v_g is the rightmost vertex in its row in Γ'_{i-1} and can see towards infinity on the right. But then v_g can also see the point $(+\infty, y(v_i))$, or in other words, there exists some X_g such that v_g can see all points $(X, y(v_i))$ for $X \geq X_g$. Impose the lower bound $X(v_i) \geq \lfloor X_g \rfloor + 1$ on the x -coordinate of v_i .

Now let $X(v_i)$ be the smallest value that satisfies the above lower bounds (from the row $y(v_i)$ and from all predecessors of v_i in different rows.) Set $X(v_i) = 1$ if there were no such lower bounds.

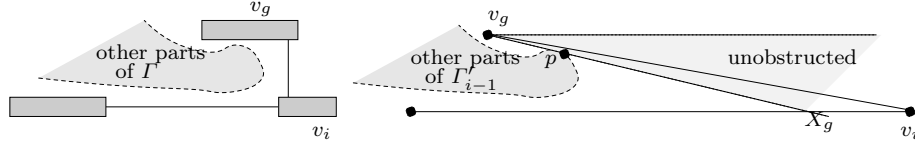


Fig. 7. Transforming a flat visibility drawing into a straight-line drawing with the same y -coordinates.

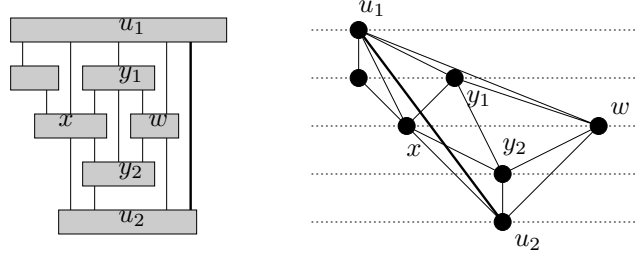


Fig. 8. An example where the algorithm from [2] creates a crossing. Edge (u_1, u_2) (which is drawn “too early”) is bold.

A.2 A bad example

Unfortunately, the above algorithm sometimes does not give planar drawings. The above algorithm preserves (as one can easily show by induction) the left-to-right-orders among vertices, but it does not necessarily preserve left-to-right-orders when we also include the points where edges cross rows.

Figure 8 shows a specific example where the algorithm goes wrong. The vertices are being placed in order $\dots, u_1, \dots, x, u_2, \dots, y_1, \dots$. When placing u_2 , we also insert the edge (u_1, u_2) , because u_1 is a predecessor of u_2 . But this edge is “much farther right” in the sense that in the second row from top, this edge should come *after* y_1 , while the algorithm draws it before placing y_1 . Consequently, the vertices $\{y_1, y_2, w\}$, which should be drawn inside triangle $\{x, u_1, u_2\}$, are instead drawn to the right of it, resulting in crossings.

In fact, on this graph the algorithm cannot preserve left-to-right-orders among edges, even if we used some different processing order v_1, \dots, v_n . For to preserve left-to-right-orders, we would have to place (x, u_1) *before* placing y_1 , and so in particular both x and u_1 must come before y_1 in the vertex order. Likewise x and u_2 must come before y_2 in the vertex order. On the other hand, w must come *after* both y_1 and y_2 due to edge (y_1, y_2) . Combining the above, we see that both u_1 and u_2 must come before w . But then edge (u_1, u_2) is drawn before placing vertex w , contradicting the left-to-right-orders of edges in w ’s row.

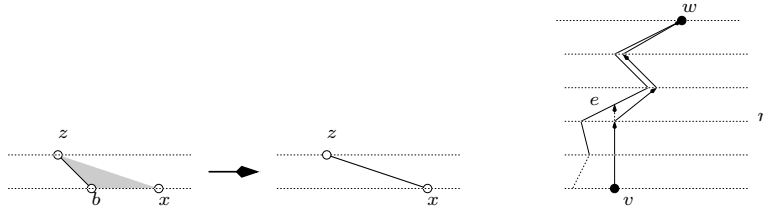


Fig. 9. Shortcutting a bend incident to a horizontal edge segment, and inserting a new path to a vertex higher up.

B Triangulating a y -monotone poly-line drawing

We now give the full details of how to triangulate a y -monotone poly-line drawing. We assume that vertices, edges and rows have already been added so that the outer-face is a triangle. Convert—by inserting bends as needed—the given drawing into a *short* y -monotone poly-line drawing, where any edge-segment is horizontal or connects two adjacent rows. Call a poly-line drawing *strictly* y -monotone if any edge is either horizontal or drawn with a strictly y -monotone path. Any short y -monotone poly-line drawing can be made strict: If some bend b is incident to a horizontal segment s_h and a non-horizontal segment s_v , then connect directly from the other end x of s_h to the other end z of s_v . Since x is one row above or below z , this cannot introduce crossings. See Figure 9.

Now we have a short strictly y -monotone poly-line drawing. If some inner vertex v has no neighbour w with $y(w) > y(v)$, then go upward from v until we hit some edge e , say at y -coordinate Y . Let w be the end of e with larger y -coordinate (breaking ties arbitrarily) and add edge (v, w) to the graph. Add a strictly y -monotone path for (v, w) to Γ as follows. Define $r := \lceil Y \rceil - 1$ to be the row just below where we hit e . Go upward from v until row r , then in rows $r + 1, \dots, y(w) - 1$ add a bend next to the bend of e (on the side from which we hit e), and then connect to w . See Figure 9. No edge can cross this path that wouldn't have crossed e , by choice of e and the placement of bends.

In this fashion add edges until any inner vertex has neighbours strictly above and below. In the resulting drawing any inner face f is a y -monotone polygon and has no horizontal edges except perhaps a single horizontal edge each at the minimum or maximum y -coordinate. If f contains four or more vertices, then it must contain two vertices v, w such that (v, w) is not an edge; otherwise we had an outer-planar drawing of K_4 . We now show how to add (v, w) to Γ .

If $y(v) \neq y(w)$, say $y(v) < y(w)$, then find points in rows $y(v) + 1, y(v) + 2, \dots, y(w) - 1$ that are strictly inside f . Since f is y -monotone, any two consecutive such points can be connected without crossing, so draw (v, w) following these points. If $y(v) = y(w)$ then the horizontal segment from v to w belongs to the closure of the y -monotone face f . In fact it lies strictly inside f since the only horizontal edge of f are single edges at the minimum and maximum y -coordinate, but (v, w) was not an edge before. So we can draw (v, w) horizontally.

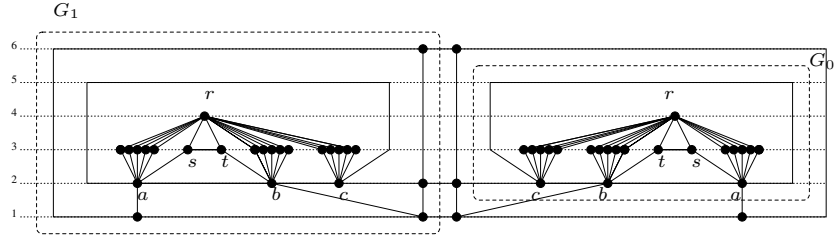


Fig. 10. A graph that can be drawn on 6 rows, but not if edges must be y -monotone.

C y -monotonicity

This section gives the full proof that y -monotonicity is restrictive, i.e., that sometimes the height can be reduced if edges are not drawn y -monotone.

Theorem 3. *There exists a graph with a planar poly-line drawing on 6 rows that has no planar y -monotone poly-line drawing on 6 rows.*

Proof. The graph G and the poly-line drawing on 6 rows is shown in Figure 10. Graph G consists of two copies of G_1 , which in turn consists of a 4-cycle that surrounds a graph G_0 . Graph G_0 is constructed from a triangular prism with triangles $\{a, b, c\}$ and $\{r, s, t\}$. The edge (c, r) of the triangular prism has been replaced by $K_{2,5}$, and two additional copies of $K_{2,5}$ have been inserted at $\{b, r\}$ and $\{a, r\}$. Observe that G has only one planar embedding up to symmetry, since its only cutting pairs are the 2-sides of each $K_{2,5}$. We need an observation.

Claim: *Any planar drawing of $K_{2,5}$ on three rows contains the 2-side vertices on the top and bottom row.* For the outer-face of $K_{2,5}$ consists of a 4-cycle, leaving three vertices (all from the 5-side) that are surrounded by a 4-cycle and hence on the middle row. By planarity this forces the vertices of the 2-side onto the bottom row and the top row.

Now assume that Γ is a y -monotone poly-line drawing of G on 6 rows. At least one copy of G_1 has the 4-cycle as outer-face in the induced drawing. Because this 4-cycle enclosing G_0 , the induced drawing of G_0 must exist entirely on rows 2,3,4,5. Because G_0 consists of a triangle $\{a, b, c\}$ enclosing another triangle, the inner triangle $\{r, s, t\}$ is on rows 3 and 4 only.

Triangle $\{a, b, c\}$ encloses a triangle on rows 3 and 4, and hence must contain points on rows 2 and 5. By y -monotonicity, the maximum and minimum y -coordinate of the triangle must be achieved at vertices, so one of the vertices, say a , has y -coordinate 2, while another one, say c , has y -coordinate 5.

Assume vertex r had y -coordinate 3. Then the $K_{2,5}$ with 2-side $\{a, r\}$ is entirely drawn on rows 2,3,4 (because all vertices except a are inside triangle $\{a, b, c\}$), but r and a are on adjacent rows. This contradicts the claim. If r has y -coordinate 4 then similarly we find a contradiction at the $K_{2,5}$ with 2-side $\{c, r\}$. So no y -monotone poly-line drawing of G on 6 rows can exist. \square